SPATIAL TRANSFORMATIONS

TRANSFORMATIONS BETWEEN THE
GEODETIC AND ASTRONOMICAL
REFERENCE & COORDINATE SYSTEMS

Mark Jones

Abstract

This document provides the algorithms for the transformation of point coordinates between a number of geodetic reference systems, astronomical reference systems, and associated coordinate systems.

Key words: Geodesy, Astronomy, Transformation, Geodetic, Astronomical, Reference System

Mots-clés: Géodésie, Astronomie, Transformation, Géodésique, Astronomique, Système de Référence
# TABLE OF CONTENTS

1. INTRODUCTION .......................................................................................................................... 1
2. CONVENTIONAL TERRESTRIAL AND GEODETIC REFERENCE SYSTEMS .................. 2
   2.1 Geodetic Cartesian to Conventional Terrestrial ................................................................. 2
   2.2 Conventional Terrestrial to Geodetic Cartesian ............................................................... 3
3. GEODETIC AND LOCAL GEODETIC REFERENCE SYSTEMS ........................................... 3
   3.1 Local Geodetic to Geodetic Cartesian .............................................................................. 3
   3.2 Geodetic Cartesian to Local Geodetic ............................................................................. 4
4. LOCAL GEODETIC AND MODIFIED LOCAL GEODETIC SYSTEMS ............................... 4
   4.1 Local Geodetic to Modified Local Geodetic .................................................................... 4
   4.2 Modified Local Geodetic to Local Geodetic .................................................................... 5
5. LOCAL GEODETIC AND LOCAL ASTRONOMICAL SYSTEMS ...................................... 5
   5.1 Local Astronomical to Local Geodetic ............................................................................ 5
   5.2 Local Geodetic to Local Astronomical .......................................................................... 6
6. LOCAL ASTRONOMICAL AND MODIFIED LOCAL ASTRONOMICAL SYSTEMS ........ 6
   6.1 Local Astronomical to Modified Local Astronomical .................................................... 6
   6.2 Modified Local Astronomical to Local Astronomical .................................................. 7
7. GEODETIC CARTESIAN AND GEODETIC ELLIPSOIDAL COORDINATE SYSTEMS .......... 7
   7.1 Geodetic Ellipsoidal to Geodetic Cartesian ..................................................................... 7
   7.2 Geodetic Cartesian to Geodetic Ellipsoidal ..................................................................... 8

Figure 1 – Transformations and Reference Systems ....................................................................... 1
1. INTRODUCTION

This document provides the algorithms for the transformation of point coordinates between a number of geodetic and astronomical reference systems, together with the conversions between their associated coordinate systems. The systems covered are those defined in [1]. The relevant transformations are to be found in [2].

The direct transformation between any two of these reference systems is not always defined\(^1\). In those cases where it is not explicitly given, the required transformation is an appropriate sequence of those direct transformations that are defined. Figure 1 shows the different reference systems and the transformations that may be established between them.

![Figure 1 – Transformations and Reference Systems](image)

For each pair of systems where a transformation is explicitly defined, the equations for the transformation in both senses are given. The transformation between these reference systems is always defined in terms of the Cartesian coordinate system associated with them, and is in the form of a Helmert Transformation [3].

\(^1\) Nevertheless, it is clear that all pairs of reference systems in Euclidean space can be related by any kind of unitary transformation, e.g. the Helmert transformation, of their Cartesian coordinates.
If necessary before any transformation is applied, geodetic ellipsoidal coordinates should first be converted to Cartesian coordinates, or following the transformation, the Cartesian coordinates should be converted to geodetic ellipsoidal coordinates.

The transformations between the Conventional Terrestrial and the Local Astronomical reference systems are not provided. They follow the same pattern as the transformations between the Geodetic and Local Geodetic reference systems.

2. CONVENTIONAL TERRESTRIAL AND GEODETIC REFERENCE SYSTEMS

A geodetic reference system may, in each instance, be located relative to a conventional terrestrial system, and oriented with respect to the latter. The transformation linking the two is a simple Helmert Transformation, without a scale factor, where the parameters are the set of geocentric datum position parameters.

The parameters involved are the datum translation components, \( x_E, y_E, z_E \), and the datum misalignment angles, \( e_x, e_y, e_z \).

If the datum is perfectly aligned and the misalignment angles are all zero, the transformation between the two systems is reduced to a translation of point coordinates.

Determining the position and orientation of a geodetic reference system is referred to as the establishment of a horizontal geodetic datum.

If this transformation is to be applied to a free vector, the translation vector should be disregarded.

2.1 Geodetic Cartesian to Conventional Terrestrial

This Helmert transformation, without a scale factor, is defined as follows:

\[
x_{CT} = R_{xyz} X_{GC} + t
\]

where,

\( x_{CT} \) = conventional terrestrial Cartesian coordinates

\( X_{GC} \) = geodetic Cartesian coordinates

\( R_{xyz} = R_x(e_x) R_y(e_y) R_z(e_z) \) = rotation matrix \[4\]

\( t = \begin{pmatrix} x_E \\ y_E \\ z_E \end{pmatrix} \) = translation vector, datum translation components
\[
\begin{bmatrix}
  e_x \\
  e_y \\
  e_z \\
\end{bmatrix}
= \text{datum misalignment angles}
\]

### 2.2 Conventional Terrestrial to Geodetic Cartesian

The inverse transformation is defined as follows:

\[
X_{GC} = R_{sys}^{-1} (x_{CT} - t)
\]  
(2)

where the notation is unchanged.

### 3. Geodetic and Local Geodetic Reference Systems

A local geodetic system\(^1\) is in each instance located relative to a geodetic reference system, and oriented with respect to the latter.

The transformation linking the two is a simple Helmert transformation, where the parameters are the geodetic coordinates of the origin of the local geodetic system. The geodetic ellipsoidal coordinates of the origin provide the orientation parameters, and the geodetic Cartesian coordinates the translation vector.

If this transformation is to be applied to a free vector, the translation vector should be disregarded.

#### 3.1 Local Geodetic to Geodetic Cartesian

As mentioned, the transformation between these two systems is a Helmert transformation defined as follows:

\[
X_{GC} = R_{3D} x_{LG} + t
\]  
(3)

where,

- \(X_{GC}\) = geodetic Cartesian coordinates
- \(x_{LG}\) = local geodetic coordinates
- \(R_{3D}\) = \(R_z(\pi - \lambda) R_y(\pi/2 - \phi) R_z(\pi/2)\) = rotation matrix \([5]\)
- \(t = X_0\) = translation vector

\(^1\) The Local Geodetic System is considered to be defined as in \([1]\), and as such is the basis for a right-handed Cartesian system. This is in contrast to the standard definition.
\[ \phi = \begin{pmatrix} \phi \\ \lambda \end{pmatrix} \] = the ellipsoidal coordinates of the origin of the local geodetic system

\[ X_0 = \begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix} \] = the geodetic Cartesian coordinates of the origin of the local geodetic system

### 3.2 Geodetic Cartesian to Local Geodetic

The inverse transformation is defined as follows:

\[ x_{LG} = R_{3D}^{-1} (X_{GC} - t) \]  \hspace{1cm} (4)

### 4. LOCAL GEODETIC AND MODIFIED LOCAL GEODETIC SYSTEMS

As defined in [1] a modified local geodetic system is derived from a local geodetic by means of a re-orientation of the axes and a translation of the origin. In effect the two systems are related by means of a Helmert transformation.

By definition the re-orientation of the local geodetic system axes involves a general 3D rotation, however, in practice this is most likely to be a simple rotation about the local z-axis. In this way either the x- or y-axis of the modified local geodetic system is defined to have a given geodetic azimuth.

If this transformation is to be applied to a free vector, the translation vector should be disregarded.

#### 4.1 Local Geodetic to Modified Local Geodetic

The transformation between these two systems is a Helmert transformation with no scale factor, defined as follows:

\[ x_{MLG} = R_{zxy} x_{LG} + t \]  \hspace{1cm} (5)

where,

\[ x_{MLG} \] = modified local geodetic Cartesian coordinates

\[ x_{LG} \] = local geodetic Cartesian coordinates

\[ R_{zxy} \] = rotation matrix [6]

\[ t \] = translation vector
4.2 Modified Local Geodetic to Local Geodetic

The inverse transformation to that defined in §4.1, is defined as follows:

\[ x_{LG} = R_{xy}^{-1} (x_{MLG} - t) \]  

(6)

where the notation is the same as §4.1.

5. LOCAL GEODETIC AND LOCAL ASTRONOMICAL SYSTEMS

These two reference systems are both topocentric and, if they have the same origin, the standard transformation between them is a rotation. The rotation accounts for the misalignment between the geodetic reference system and the conventional terrestrial system, together with the difference between the geodetic coordinates and astronomical coordinates of the origin. These combine to give the deviation of the vertical at the origin (the difference in direction between the ellipsoid normal and the gravity vector) and an azimuth correction, the parameters of the rotation matrix. The geoid model at CERN [7] allows us to derive these same parameters.

However, we have not adopted the standard definition for these two reference systems\(^1\) since we have defined reference systems forming the basis for right-handed Cartesian coordinate systems. It is therefore necessary to obtain the left-handed Cartesian coordinates before applying the rotation matrix, and from the resulting coordinates derive the right-handed Cartesian coordinates.

5.1 Local Astronomical to Local Geodetic

To obtain the left-handed local geodetic Cartesian coordinates we initially apply a reflection in the x = y plane. The rotation matrix is then expressed as three rotations, one about each axis of the coordinate system. To obtain the right-handed local astronomical Cartesian coordinates we once again apply a reflection in the x = y plane. The transformation is defined as follows:

\[ x_{LG} = P_{xy} R_{ox} P_{xy} x_{LA} \]  

(7)

where,

- \( x_{LG} \) = local geodetic Cartesian coordinates
- \( x_{LA} \) = local astronomical Cartesian coordinates
- \( P_{xy} \) = reflection matrix [8]
- \( R_{ox} = R_x (\Delta \Lambda) R_y (-\xi) R_x (\eta) \) = rotation matrix [6]

\(^1\) The Local Geodetic and Local Astronomical Systems are considered to be defined as in [1], and as such are the basis for right-handed Cartesian systems. This is in contrast to the standard definitions.
and,

\[
\begin{bmatrix}
\Delta A \\
\xi \\
-\eta
\end{bmatrix} = \begin{bmatrix}
(\Lambda - \lambda) \sin \phi \\
\Phi - \phi \\
-(\Lambda - \lambda) \cos \phi
\end{bmatrix}
- R_y (\phi - \pi) R_z (\lambda - \pi)
\begin{bmatrix}
e_x \\
e_y \\
e_z
\end{bmatrix}
\]  \hspace{1cm} (8)

\[
\begin{bmatrix}
\Phi \\
\Lambda
\end{bmatrix} = \text{astronomical coordinates of origin}
\]

\[
\begin{bmatrix}
\phi \\
\lambda
\end{bmatrix} = \text{geodetic ellipsoidal coordinates of origin}
\]

\[
\begin{bmatrix}
e_x \\
e_y \\
e_z
\end{bmatrix} = \text{datum misalignment angles, see §2}
\]

5.2 **Local Geodetic to Local Astronomical**

The inverse transformation is defined quite simply as follows:

\[
x_{LA} = P_y R^{-1}_{yx} P_y x_{LG}
\]  \hspace{1cm} (9)

where the notation is unchanged.

6. **LOCAL ASTRONOMICAL AND MODIFIED LOCAL ASTRONOMICAL SYSTEMS**

As defined in [1] a modified local astronomical system is derived from a local astronomical by means of a re-orientation of the axes and a translation of the origin. In effect the two systems are related by means of a Helmert transformation.

By definition the re-orientation of the local astronomical system axes involves a general 3D rotation, however, in practice this is most likely to be a simple rotation about the local z-axis. In this way either the x- or y-axis of the modified local astronomical system is defined to have a given astronomical azimuth.

If this transformation is to be applied to a free vector, the translation vector should be disregarded.

6.1 **Local Astronomical to Modified Local Astronomical**

The transformation between these two systems is a Helmert transformation with no scale factor, defined as follows:

\[
x_{MLA} = R_{yx} x_{LA} + t
\]  \hspace{1cm} (10)
where,

\[ x_{\text{MLA}} = \text{modified local astronomical Cartesian coordinates} \]

\[ x_{\text{LA}} = \text{local astronomical Cartesian coordinates} \]

\[ R_{\eta x} = \text{rotation matrix [6]} \]

\[ t = \text{translation vector} \]

### 6.2 Modified Local Astronomical to Local Astronomical

The inverse transformation to that defined in §6.1, is defined as follows:

\[
x_{\text{LA}} = R_{\eta x}^{-1} (x_{\text{MLA}} - t)
\]

where the notation is the same as §6.1.

### 7. Geodetic Cartesian and Geodetic Ellipsoidal Coordinate Systems

Here we deal with the conversion of point coordinates between two coordinate systems defined with respect to the same geodetic reference system.

#### 7.1 Geodetic Ellipsoidal to Geodetic Cartesian

The conversion from the geodetic ellipsoidal coordinate system to the geodetic Cartesian coordinate system may be expressed explicitly by the following equations:

\[
X_{\text{GC}} = \begin{pmatrix} X_{\text{GC}} \\ Y_{\text{GC}} \\ Z_{\text{GC}} \end{pmatrix} = \begin{pmatrix} (\nu + h) \cos \phi \cos \lambda \\ (\nu + h) \cos \phi \sin \lambda \\ (b^2 / a^2 + h) \sin \phi \end{pmatrix}
\]

where,

\[
\phi = \begin{pmatrix} \phi \\ \lambda \\ h \end{pmatrix} = \text{geodetic ellipsoidal coordinates}
\]

\[ \nu = \text{prime vertical radius of curvature at } \phi \]

\[ a = \text{ellipsoid semi-major axis} \]

\[ b = \text{ellipsoid semi-minor axis} \]
For further explanation of these terms see [1].

7.2 Geodetic Cartesian to Geodetic Ellipsoidal

The conversion between these two coordinate systems in the reverse sense is not so straightforward. A number of different algorithms are available for determining the geodetic ellipsoidal coordinates, see [2]. Although non-iterative algorithms do exist, the algorithm presented here is an iterative process, which gives good control over the precision retained in the transformed coordinates. The algorithm determines, by iteration, the values of, $\phi$, and, $h$, until there are only changes below a specified tolerance level.

The algorithm is defined by letting:

\[
\begin{align*}
  h_{i+1} &= \frac{p}{\cos \phi_i} - \nu_i \\
  \phi_{i+1} &= \arctan \left( \frac{Z_{GC}}{p \left( 1 - e^2 \hat{Q}_i \right)} \right)
\end{align*}
\]

where,

\[
\nu_i = \nu(\phi_i) \quad \text{and} \quad p = \sqrt{X_{GC}^2 + Y_{GC}^2}
\]

These equations are solved iteratively starting from the initial values,

\[
\begin{align*}
  h_0 &= 0 \\
  \phi_0 &= \arctan \left( \frac{Z_{GC}}{p(1-e^2)} \right)
\end{align*}
\]

The iterative process is stops when the change in the values of, $\phi$, and, $h$, are sufficiently small for the precision of the calculation,

\[
|h_{i+1} - h_i| < e_h
\]

\[
|\phi_{i+1} - \phi_i| < e_\phi
\]

The value of lambda is determined from,

\[
\lambda = 2 \arctan \left( \frac{Y_{GC}}{X_{GC} + p} \right)
\]
References


